Differential operators on Schwartz distributions. Jet formalism

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Differential operators on Schwartz distributions conventionally are defined as the transpose of differential operators on functions with compact support. They do not exhaust all differential operators. We follow algebraic formalism of differential operators on modules over commutative rings. In a general setting, Schwartz distributions on sections with compact support of vector bundles on an arbitrary smooth manifold are considered.

1 Introduction

Quantum field theory provides examples of differential operators and differential equations on distributions.

Let U be an open subset of $\mathbb{R}^n$, $\mathcal{D}(U)$ the space of smooth real functions with compact support, and $\mathcal{D}(U)'$ the space of Schwartz distribution on $\mathcal{D}(U)$. Differential operators on $\mathcal{D}(U)'$ conventionally are defined as the transpose of differential operators on $\mathcal{D}(U)$ [2]. However, they do not exhaust all differential operators on $\mathcal{D}(U)'$.

We follow algebraic formalism of differential operators on modules over commutative rings and jets of modules (Section 2) [3, 7].

In a general setting, Schwartz distributions on sections with compact support of a vector bundles Y on an arbitrary smooth manifold X are considered. We follow familiar formalism of distributions, not the nonlinear ones [8].

Let $Y(X)$ denote a $C^\infty(X)$-module of global sections of $Y \to X$. Let $E \to X$ be a vector bundle. The $E(X)$-valued differential operators defined on $Y(X)$ as a $C^\infty(X)$-module coincide with the familiar ones (Section 3). They constitute the $C^\infty(X)$-module (32).

Let $\mathcal{D}(Y)$ be a $C^\infty(X)$-module of sections with compact support of $Y \to X$. It is provided with a LF-topology similar to that on $\mathcal{D}(U)$. With this topology, $\mathcal{D}(U)$ is a nuclear vector space and a topological $C^\infty(X)$-module. Differential operators on $\mathcal{D}(Y)$ as a $C^\infty(X)$-module are defined (Section 4). We show that that there is one-to-one correspondence between $E(X)$-valued differential operators on $\mathcal{D}(Y)$ and those on $Y(X)$ (Theorem 11), and that $\mathcal{D}(Y)$-valued differential operators on $\mathcal{D}(Y)$ are continuous (Theorem 13).

Let $\mathcal{D}(Y)'$ be the topological dual of $\mathcal{D}(Y)$ endowed with the strong topology. It is a nuclear vector space whose topological dual is $\mathcal{D}(Y)$, and it is a topological $C^\infty(X)$-module. Differential operators on $\mathcal{D}(Y)'$ as a $C^\infty(X)$-module are defined. In particular, the transpose of any $\mathcal{D}(Y)$-valued differential operator on $\mathcal{D}(Y)$ is a differential operator on $\mathcal{D}(Y)'$ (Theorem 15). However, a differential operator on $\mathcal{D}(Y')$ need not be of this

type. We show that a $\mathcal{D}(Y)'$-valued differential operator on $\mathcal{D}(Y)'$ is the transpose of a $\mathcal{D}(Y)$-valued differential operator on $\mathcal{D}(Y)$ iff it is continuous (Theorem 16).

2 Differential operators on modules

This Section summarizes the relevant material on differential operator on modules over a commutative ring [3, 7, 10].

Let $\mathcal{K}$ be a commutative ring (i.e., a commutative unital algebra) and $\mathcal{A}$ a commutative $\mathcal{K}$-ring. Let P and Q be $\mathcal{A}$-modules. The $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ of $\mathcal{K}$-homomorphisms $\Phi : P \to Q$ can be endowed with the two different $\mathcal{A}$-module structures

$$(a\Phi)(p) = a\Phi(p), \qquad (a \bullet \Phi)(p) = \Phi(ap), \qquad a \in \mathcal{A}, \quad p \in P. \tag{1}$$

We refer to the second one as a $\mathcal{A}^\bullet$-module structure. Let us put

$$\delta_a\Phi = a\Phi - a \bullet \Phi, \qquad a \in \mathcal{A}. \tag{2}$$

Definition 1: An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is called a k-order Q-valued differential operator on P if

$$\delta_{a_0} \circ \cdots \circ \delta_{a_k}\Delta = 0$$

for any tuple of $k+1$ elements $a_0,\ldots,a_k$ of $\mathcal{A}$. The set $\mathrm{Diff}_k(P,Q)$ of these operators inherits the $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures (1).

In particular, zero order differential operators are $\mathcal{A}$-homomorphisms $P \to Q$. A first order differential operator Δ satisfies the condition

$$\delta_b \circ \delta_a \Delta(p) = ba\Delta(p) - b\Delta(ap) - a\Delta(bp) + \Delta(abp) = 0, \quad a, b \in \mathcal{A}.$$

Let $P = \mathcal{A}$. Any zero order Q-valued differential operator Δ on $\mathcal{A}$ is defined by its value $\Delta(\mathbf{1})$. Then there is an isomorphism

$$\mathrm{Diff}_0(\mathcal{A}, Q) = Q$$

via the association

$$Q \ni q \to \Delta_q \in \mathrm{Diff}_0(\mathcal{A}, Q), \qquad \Delta_q(\mathbf{1}) = q,$$

A first order Q-valued differential operator Δ on $\mathcal{A}$ fulfils the condition

$$\Delta(ab) = b\Delta(a) + a\Delta(b) - ba\Delta(\mathbf{1}), \qquad a, b \in \mathcal{A}.$$

It is a Q-valued derivation of $\mathcal{A}$ if $\Delta(\mathbf{1}) = 0$, i.e., the Leibniz rule

$$\Delta(ab) = \Delta(a)b + a\Delta(b), \qquad a, b \in \mathcal{A}, \tag{3}$$

holds. Any first order differential operator on $\mathcal{A}$ falls into the sum

$$\Delta(a) = a\Delta(\mathbf{1}) + [\Delta(a) - a\Delta(\mathbf{1})]$$

of a zero order differential operator and a derivation. Accordingly, there is an $\mathcal{A}$-module decomposition

$$\mathrm{Diff}\,_1(\mathcal{A}, Q) = Q \oplus \mathfrak{d}(\mathcal{A}, Q), \tag{4}$$

where $\mathfrak{d}(\mathcal{A}, Q)$ is an $\mathcal{A}$-module of Q-valued derivations of $\mathcal{A}$.

If $P = Q = \mathcal{A}$, the derivation module $\mathfrak{d}\mathcal{A}$ of $\mathcal{A}$ is a Lie $\mathcal{K}$-algebra. Accordingly, the decomposition (4) takes the form

$$\mathrm{Diff}\,_1(\mathcal{A}) = \mathcal{A} \oplus \mathfrak{d}\mathcal{A}. \tag{5}$$

Example 1: Let X be an n-dimensional real smooth manifold coordinated by x^μ, and let $C^\infty(X)$ be an $\mathbb{R}$-ring of smooth real functions on X. There is one-to-one correspondence between the derivations of $C^\infty(X)$ and the vector fields on X. It is given by the expression

$$T(X) = TX(X) \ni u \leftrightarrow \mathbf{L}_u \in \mathfrak{d}C^\infty(X), \qquad \mathbf{L}_u(f) = u^\mu \partial_\mu f, \qquad f \in C^\infty(X), \tag{6}$$

where $\mathbf{L}_u$ denotes the Lie derivative along u.

The study of Q-valued differential operators on an $\mathcal{A}$-module P is reduced to that of Q-valued differential operators on a ring $\mathcal{A}$ as follows.

Theorem 2: Let us consider an $\mathcal{A}$-homomorphism

$$h_k : \mathrm{Diff}\,_k(\mathcal{A}, Q) \to Q, \qquad h_k(\Delta) = \Delta(\mathbf{1}). \tag{7}$$

Any k-order Q-valued differential operator $\Delta \in \mathrm{Diff}\,_k(P, Q)$ on P uniquely factorizes as

$$\Delta : P \xrightarrow{\mathfrak{f}_\Delta} \mathrm{Diff}\,_k(\mathcal{A}, Q) \xrightarrow{h_k} Q \tag{8}$$

through the homomorphism h_k (7) and some homomorphism

$$\mathfrak{f}_\Delta : P \to \mathrm{Diff}\,_k(\mathcal{A}, Q), \qquad (\mathfrak{f}_\Delta p)(a) = \Delta(ap), \qquad a \in \mathcal{A}, \tag{9}$$

of an $\mathcal{A}$-module P to an $\mathcal{A}^\bullet$-module $\mathrm{Diff}\,_k(\mathcal{A}, Q)$. The assignment $\Delta \to \mathfrak{f}_\Delta$ defines an $\mathcal{A}^\bullet - \mathcal{A}$-module isomorphism

$$\mathrm{Diff}\,_k(P, Q) = \mathrm{Hom}\,_{\mathcal{A}-\mathcal{A}^\bullet}(P, \mathrm{Diff}\,_k(\mathcal{A}, Q)). \tag{10}$$

In a different way, k-order differential operators on a module P are represented by zero order differential operators on a module of k-order jets of P as follows.

Given an $\mathcal{A}$-module P, let us consider a tensor product $\mathcal{A} \otimes_\mathcal{K} P$ of $\mathcal{K}$-modules $\mathcal{A}$ and P. We put

$$\delta^b(a \otimes p) = (ba) \otimes p - a \otimes (bp), \qquad p \in P, \qquad a, b \in \mathcal{A}. \tag{11}$$

Let us denote by μ^{k+1} a submodule of $\mathcal{A}\otimes_{\mathcal{K}} P$ generated by elements of the type

$$\delta^{b_0}\circ\cdots\circ\delta^{b_k}(a\otimes p).$$

Definition 3: A k-order jet module $\mathcal{J}^k(P)$ of a module P is the quotient of the $\mathcal{K}$-module $\mathcal{A}\otimes_{\mathcal{K}} P$ by μ^{k+1}. We denote its elements $c\otimes_k p$.

In particular, a first order jet module $\mathcal{J}^1(P)$ is generated by elements $\mathbf{1}\otimes_1 p$ modulo the relations

$$\delta^a\circ\delta^b(\mathbf{1}\otimes_1 p) = ab\otimes_1 p - b\otimes_1 (ap) - a\otimes_1 (bp) + \mathbf{1}\otimes_1 (abp) = 0. \tag{12}$$

A $\mathcal{K}$-module $\mathcal{J}^k(P)$ is endowed with the $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures

$$b(a\otimes_k p) = ba\otimes_k p, \qquad b\bullet(a\otimes_k p) = a\otimes_k (bp). \tag{13}$$

There exists a homomorphism

$$J^k : P\ni p\to \mathbf{1}\otimes_k p\in \mathcal{J}^k(P) \tag{14}$$

of an $\mathcal{A}$-module P to an $\mathcal{A}^\bullet$-module $\mathcal{J}^k(P)$ such that $\mathcal{J}^k(P)$, seen as an $\mathcal{A}$-module, is generated by elements J^kp, $p\in P$.

Due to the natural monomorphisms $\mu^r\to\mu^k$ for all $r>k$, there are $\mathcal{A}$-module epimorphisms of jet modules

$$\pi^{i+1}_i : \mathcal{J}^{i+1}(P)\to \mathcal{J}^i(P).$$

In particular,

$$\pi^1_0 : \mathcal{J}^1(P)\ni a\otimes_1 p\to ap\in P. \tag{15}$$

Theorem 4: Any k-order Q-valued differential operator Δ on an $\mathcal{A}$-module P factorizes uniquely

$$\Delta : P\xrightarrow{J^k}\mathcal{J}^k(P)\xrightarrow{\mathfrak{f}^\Delta} Q$$

through the homomorphism J^k (14) and some $\mathcal{A}$-homomorphism $\mathfrak{f}^\Delta : \mathcal{J}^k(P)\to Q$. The association $\Delta\to\mathfrak{f}^\Delta$ yields an $(\mathcal{A}^\bullet - \mathcal{A})$-module isomorphism

$$\mathrm{Diff}\,_k(P,Q) = \mathrm{Hom}\,_{\mathcal{A}}(\mathcal{J}^k(P),Q). \tag{16}$$

Let us consider jet modules $\mathcal{J}^k=\mathcal{J}^k(\mathcal{A})$ of a ring $\mathcal{A}$ itself. In particular, the first order jet module $\mathcal{J}^1$ consists of the elements $a\otimes_1 b$, $a,b\in\mathcal{A}$, subject to the relations

$$ab\otimes_1\mathbf{1} - b\otimes_1 a - a\otimes_1 b + \mathbf{1}\otimes_1 (ab) = 0. \tag{17}$$

The $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures (13) on $\mathcal{J}^1$ read

$$c(a\otimes_1 b)=(ca)\otimes_1 b, \qquad c\bullet(a\otimes_k b)=a\otimes_1(cb)=(a\otimes_1 b)c.$$

Theorems 2 and 4 are completed with forthcoming Theorem 5 so that any one of them is a corollary of the others.

Theorem 5: There is an isomorphism

$$\mathcal{J}^k(P)=\mathcal{J}^k\underset{\mathcal{A}^\bullet-\mathcal{A}}{\otimes}P, \qquad (a\otimes_k bp)\leftrightarrow(a\otimes_1 b)\otimes p. \tag{18}$$

Then we have the $(\mathcal{A}-\mathcal{A}^\bullet)$-module isomorphisms

$$\begin{aligned}&\mathrm{Hom}\,_{\mathcal{A}}(\mathcal{J}^k\otimes P,Q)=\mathrm{Hom}\,_{\mathcal{A}-\mathcal{A}^\bullet}(P,\mathrm{Hom}\,_{\mathcal{A}}(\mathcal{J}^k,Q))=\\ &\qquad \mathrm{Hom}\,_{\mathcal{A}-\mathcal{A}^\bullet}(P,\mathrm{Diff}\,_k(\mathcal{A},Q))=\mathrm{Diff}\,_k(P,Q)=\mathrm{Hom}\,_{\mathcal{A}}(\mathcal{J}^k(P),Q).\end{aligned}$$

Besides the monomorphism (14):

$$J^1:\mathcal{A}\ni a\to\mathbf{1}\otimes_1 a\in\mathcal{J}^1,$$

there exists an $\mathcal{A}$-module monomorphism

$$i_1:\mathcal{A}\ni a\to a\otimes_1\mathbf{1}\in\mathcal{J}^1.$$

With these monomorphisms, we have the canonical $\mathcal{A}$-module splitting

$$\mathcal{J}^1=i_1(\mathcal{A})\oplus\mathcal{O}^1, \qquad J^1(b)=\mathbf{1}\otimes_1 b=b\otimes_1\mathbf{1}+(\mathbf{1}\otimes_1 b-b\otimes_1\mathbf{1}), \tag{19}$$

where the $\mathcal{A}$-module $\mathcal{O}^1$ is generated by elements $\mathbf{1}\otimes_1 b-b\otimes_1\mathbf{1}$ for all $b\in\mathcal{A}$. Let us consider a $\mathcal{K}$-homomorphism

$$d^1:\mathcal{A}\ni b\to\mathbf{1}\otimes_1 b-b\otimes_1\mathbf{1}\in\mathcal{O}^1. \tag{20}$$

This is a $\mathcal{O}^1$-valued derivation of a $\mathcal{K}$-ring $\mathcal{A}$ which obeys the Leibniz rule

$$d^1(ab)=\mathbf{1}\otimes_1 ab-ab\otimes_1\mathbf{1}+a\otimes_1 b-a\otimes_1 b=ad^1b+(d^1a)b.$$

It follows from the relation (17) that $ad^1b=(d^1b)a$ for all $a,b\in\mathcal{A}$. Thus, seen as an $\mathcal{A}$-module, $\mathcal{O}^1$ is generated by elements d^1a for all $a\in\mathcal{A}$.

Let $\mathcal{O}^{1*}=\mathrm{Hom}\,_{\mathcal{A}}(\mathcal{O}^1,\mathcal{A})$ be the dual of an $\mathcal{A}$-module $\mathcal{O}^1$. In view of the splittings (5) and (19), the isomorphism (16) leads to the duality relation

$$\mathfrak{d}\mathcal{A}=\mathcal{O}^{1*}, \qquad \mathfrak{d}\mathcal{A}\ni u\leftrightarrow\phi_u\in\mathcal{O}^{1*}, \qquad \phi_u(d^1a)=u(a), \qquad a\in\mathcal{A}. \tag{21}$$

Example 2: If $\mathcal{A} = C^\infty(X)$ in Example 1, then $\mathcal{O}^1 = \mathcal{O}^1(X)$ is a module of differential one-forms on X, and there is an isomorphism $\mathcal{O}^1(X) = T(X)^*$, besides the isomorphism $T(X) = \mathcal{O}^1(X)^*$ (21).

Let us return to the first order jet module $\mathcal{J}^1(P)$ of an $\mathcal{A}$-module P. Due to the isomorphism (18), the isomorphism (19) leads to the splitting

$$\begin{aligned} &\mathcal{J}^1(P) = (\mathcal{A}\oplus\mathcal{O}^1) \underset{\mathcal{A}^\bullet-\mathcal{A}}{\otimes} P = (\mathcal{A}\underset{\mathcal{A}}{\otimes} P)\oplus(\mathcal{O}^1 \underset{\mathcal{A}^\bullet-\mathcal{A}}{\otimes} P), \\ &a\otimes_1 bp \leftrightarrow (ab + ad^1(b))\otimes p. \end{aligned} \tag{22}$$

Applying the epimorphism π^1_0 (15) to this splitting, one obtains the short exact sequence of $(\mathcal{A}-\mathcal{A}^\bullet)$-modules

$$\begin{aligned} &0\longrightarrow \mathcal{O}^1 \underset{\mathcal{A}^\bullet-\mathcal{A}}{\otimes} P \to \mathcal{J}^1(P) \xrightarrow{\pi^1_0} P \longrightarrow 0, \\ &(a\otimes_1 b - ab\otimes_1 \mathbf{1})\otimes p \to (c\otimes_1 \mathbf{1} + a\otimes_1 b - ab\otimes_1 \mathbf{1})\otimes p \to cp. \end{aligned} \tag{23}$$

It is canonically split by the $\mathcal{A}^\bullet$-homomorphism

$$P\ni ap \to \mathbf{1}\otimes_1 ap = a\otimes_1 p + d^1(a)\otimes_1 p \in \mathcal{J}^1(P).$$

However, it need not be split by an $\mathcal{A}$-homomorphism, unless P is a projective $\mathcal{A}$-module.

Definition 6: A connection on an $\mathcal{A}$-module P is defined as an $\mathcal{A}$-homomorphism

$$\Gamma : P \to \mathcal{J}^1(P), \qquad \Gamma(ap) = a\Gamma(p), \tag{24}$$

which splits the exact sequence (23).

Given the splitting Γ (24), let us define a complementary morphism

$$\nabla = J^1 - \Gamma : P \to \mathcal{O}^1 \underset{\mathcal{A}^\bullet-\mathcal{A}}{\otimes} P, \qquad \nabla(p) = \mathbf{1}\otimes_1 p - \Gamma(p). \tag{25}$$

This also is called a connection though it in fact is a covariant differential on a module P. This morphism satisfies the Leibniz rule

$$\nabla(ap) = d^1 a\otimes p + a\nabla(p), \tag{26}$$

i.e., ∇ is first order $(\mathcal{O}^1\otimes P)$-valued differential operator on P. Thus, we come to the equivalent definition of a connection [6].

Definition 7: A connection on an $\mathcal{A}$-module P is a $\mathcal{K}$-homomorphism ∇ (25) which obeys the Leibniz rule (26).

In view of the isomorphism (21), any connection in Definition 7 determines a connection in the following sense.

Definition 8: A connection on an $\mathcal{A}$-module P is an $\mathcal{A}$-homomorphism

$$\nabla : \mathfrak{d}\mathcal{A} \ni u \to \nabla_u \in \mathrm{Diff}_1(P,P) \tag{27}$$

such that, for each $u \in \mathfrak{d}\mathcal{A}$, the first order differential operator ∇_u obeys the Leibniz rule

$$\nabla_u(ap) = u(a)p + a\nabla_u(p), \quad a \in \mathcal{A}, \quad p \in P. \tag{28}$$

Definitions 7 and 8 are equivalent if $\mathcal{O}^1 = \mathfrak{d}\mathcal{A}^*$. For instance, this is the case of $\mathcal{A} = C^\infty(X)$ in Example 2.

In particular, let P be a commutative $\mathcal{A}$-algebra and $\mathfrak{d}P$ the derivation module of P as a $\mathcal{K}$-algebra. The $\mathfrak{d}P$ is both a P- and $\mathcal{A}$-modules. Then Definition 8 is modified as follows.

Definition 9: A connection on an $\mathcal{A}$-algebra P is an $\mathcal{A}$-homomorphism

$$\nabla : \mathfrak{d}\mathcal{A} \ni u \to \nabla_u \in \mathfrak{d}P \subset \mathrm{Diff}_1(P, P), \tag{29}$$

which is a connection on P as an $\mathcal{A}$-module, i.e., it obeys the Leinbniz rule (28).

For instance, if P is an ideal of $\mathcal{A}$, there is a unique canonical connection $u \to \nabla_u = u$ on P.

3 Differential operators on sections of a vector bundle

Let X be a smooth manifold which is customarily assumed to be Hausdorff and second-countable (i.e., it has a countable base for topology). Consequently, it has a locally compact space which is a union of a countable number of compact subsets, a separable space, a paracompact and completely regular space. Let X be connected and oriented.

Let $Y \to X$ be a vector bundle over X. Its global sections s constitute a $C^\infty(X)$-module $Y(X)$.

Let J^kY be a k-order jet manifold of Y whose elements are k-order jets of sections s of $Y \to X$. It is a vector bundle $J^kY \to X$ over X. There is a $C^\infty(X)$-module isomorphism

$$\mathcal{J}^k(Y(X)) = J^kY(X) \tag{30}$$

of a k-order jet module $\mathcal{J}^k(Y(X))$ of $Y(X)$ and a module $J^kY(X)$ of global sections of a k-order jet bundle $J^kY \to X$ of $Y \to X$ [4, 9].

Let $E \to X$ be a vector bundle and $E(X)$ a $C^\infty(X)$-module of global sections of E. By virtue of Theorem 4, there is the $C^\infty(X)$-module isomorphism (16):

$$\mathrm{Diff}_k(Y(X), E(X)) = \mathrm{Hom}_{C^\infty(X)}(\mathcal{J}^k(Y(X)), E(X)) \tag{31}$$

of the module $\mathrm{Diff}_k(Y(X), E(X))$ of k-order $E(X)$-valued differential operators on $Y(X)$ and the module $\mathrm{Hom}_{C^\infty(X)}(\mathcal{J}^k(Y(X)), E(X))$ of $C^\infty(X)$-homomorphisms of $\mathcal{J}^k(Y(X))$ to $E(X)$. Since $\mathcal{J}^k(Y(X))$ (30) is a projective $C^\infty(X)$-module of finite rank, the isomorphism (31) takes the form

$$\begin{gathered}\mathrm{Diff}_k(Y(X), E(X)) = \mathrm{Hom}_{C^\infty(X)}(\mathcal{J}^k(Y(X)), E(X)) = \\ \mathcal{J}^k(Y(X))^* \underset{C^\infty(X)}{\otimes} E(X) = ((J^kY)^* \otimes E)(X).\end{gathered} \tag{32}$$

It follows that there is one-to-one correspondence between the k-order $E(X)$-valued differential operators on $Y(X)$ and the global sections of the vector bundle $(J^kY)^*\otimes E$ where $(J^kY)^*$ is the dual of a vector bundle $J^kY\to X$.

In particular, let $E=Y$. In accordance with Definition 8, a connection ∇ on a module $Y(X)$ is a $C^\infty(X)$-homomorphism

$$\nabla : \mathfrak{d}C^\infty(X)=T(X)\ni u\to \nabla_u\in \mathrm{Diff}_1(Y(X),Y(X)) \tag{33}$$

such that, for each vector field u on X, a first order differential operator ∇_u obeys the Leibniz rule (28):

$$\nabla_u(fs)=(\mathbf{L}_uf)s+f\nabla_us, \qquad s\in Y(X), \qquad f\in C^\infty(X).$$

There is one-to-one correspondence between the connections ∇^Γ on a module $Y(X)$ and the linear connections Γ on a vector bundle $Y\to X$ such that ∇^Γ is the covariant differential with respect to Γ [4, 9].

For instance, let

$$R=X\times\mathbb{R}\to X \tag{34}$$

be a trivial bundle. Its global sections are smooth real functions on X, i.e., $C^\infty(X)=R(X)$. A k-order jet manifold of this bundle is diffeomorphic to a Whitney sum

$$J^kR=R\oplus T^*X\oplus\cdots\oplus\overset{k}{\vee}T^*X \tag{35}$$

of symmetric products of the cotangent bundle T^*X to X. By virtue of the isomorphism (30), a k-order jet module $\mathcal{J}^k(C^\infty(X))=\mathcal{J}^k(R(X))$ of a ring $C^\infty(X)$ is a module of sections $J^kR(X)$ of the vector bundle $J^kR\to X$ (35). Let $E\to X$ be a vector bundle. Then the module $\mathrm{Diff}_k(C^\infty(X),E(X))$ of $E(X)$-valued differential operators on $C^\infty(X)$ is isomorphic to a module of sections of the vector bundle $(J^kR)^*\otimes E$ where

$$(J^kR)^*=R\oplus TX\oplus\cdots\oplus\overset{k}{\vee}TX \tag{36}$$

is the dual of the vector bundle (35). Thus, we have

$$\mathrm{Diff}_k(C^\infty(X),E(X))=((J^kR)^*\otimes E)(X). \tag{37}$$

In particular, let $E=R$. Then the module of k-order $C^\infty(X)$-valued differential operators on $C^\infty(X)$ is

$$\mathrm{Diff}_k(C^\infty(X)=(J^kR)^*(X). \tag{38}$$

For instance, the module $\mathrm{Diff}_1(C^\infty(X))$ of first order differential operators on $C^\infty(X)$ is isomorphic to $C^\infty(X)\oplus T(X)$ in accordance with the decomposition (5).

4 Differential operators on sections with compact support

Let us consider a $C^\infty(X)$-module $\mathcal{D}(Y) \subset Y(X)$ of sections with compact support of a vector bundle $Y \to X$. It is endowed with the following topology [1].

Let $J^\infty Y$ be the topological inductive limit of $J^k Y$, $k \in \mathbb{N}$ which is a Fréchet (not smooth) manifold [4, 9]. It is a topological vector bundle

$$\pi^\infty_0 : J^\infty Y \to X.$$

There is a certain class $\mathcal{Q}^0_\infty Y$ of real functions on $J^\infty Y$ called the smooth functions on $J^\infty Y$. Given a function $f \in \mathcal{Q}^0_\infty Y$ and a point $z \in J^\infty Y$, there exists an open neighborhood U of z such that $f|_U$ is the pull-back of a smooth function on some finite order jet manifold $J^k Y$.

Let $\mathcal{F}_\infty Y \subset \mathcal{Q}^0_\infty Y$ denote a subset of smooth functions ϕ on $J^\infty Y$ which are of finite jet order $[\phi(K)]$ on a subset $(\pi^\infty_0)^{-1}(K) \subset J^\infty Y$ over any compact subset $K \subset X$, and which are linear on fibres of $J^\infty Y \to X$. With $\phi \in \mathcal{F}_\infty Y$, one can define a seminorm

$$p_\phi(s) = \sup_{x\in X} |J^\infty s^* \phi| \tag{39}$$

on $\mathcal{D}(Y)$ where $J^\infty s$ denotes the jet prolongation of a section s to a section of $J^\infty Y \to X$. The seminorm (39) is well defined because

$$(J^\infty s^* \phi) = (J^{[\phi(\mathrm{supp}\, s)]} s^* \phi)(x)$$

is a smooth function with compact support on X.

The set of seminorms $p_\phi(s)$ (39) for all functions $\phi \in \mathcal{F}_\infty Y$ on $J^\infty Y$ yields a locally convex topology on $\mathcal{D}(Y)$ called the LF-topology. With this topology, $\mathcal{D}(Y)$ is a nuclear complete reflexive vector space, an inductive limit of a countable family of separable Fréchet spaces [11].

If $Y = R$, the $\mathcal{D}(R) = \mathcal{D}(X)$ is the well-known nuclear space of test functions on a manifold X. There is a $C^\infty(X)$-module isomorphism

$$\mathcal{D}(Y) = Y(X) \underset{C^\infty(X)}{\otimes} \mathcal{D}(X) \subset Y(X) \underset{C^\infty(X)}{\otimes} Y(X) = Y(X). \tag{40}$$

Lemma 10: Any global section σ of the dual vector bundle $Y^* \to X$ yields a continuous homomorphism of the topological vector spaces

$$\sigma : \mathcal{D}(Y) \ni s \to (s, \sigma) \in \mathcal{D}(X). \tag{41}$$

Proof: Let p_F (39) be a seminorm on $\mathcal{D}(X)$ where $F \in \mathcal{F}_\infty R$. A global section σ of $Y^* \to X$ defines a bundle morphism $\sigma : Y \to R$ over X possessing a jet prolongation

$$J^\infty \sigma : J^\infty Y \underset{X}{\longrightarrow} J^\infty R.$$

Then we have a smooth real function

$$F_\sigma = F\circ J^\infty\sigma : J^\infty Y \underset{X}{\longrightarrow} J^\infty R \to \mathbb{R}, \tag{42}$$
$$F_\sigma(z) = (F\circ J^{[F(\pi^\infty_0(z))]}\sigma)(z), \qquad z\in J^\infty Y,$$

on $J^\infty Y$ which belongs to $\mathcal{F}_\infty Y$. The function F_σ (42) yields the seminorm p_{F_σ} (39) on $\mathcal{D}(Y)$ such that

$$p_F((s,\sigma)) = p_{F_\sigma}(s)$$

for all $s\in\mathcal{D}(Y)$. It follows that $p_F((s,\sigma)) < \varepsilon$ iff $p_{F_\sigma}(s) < \varepsilon$ and, consequently, σ (41) is an open continuous map.

In the case of $Y=R$, the homomorphism (41) is a multiplication

$$g:\mathcal{D}(X)\ni f\to gf\in\mathcal{D}(X), \qquad g\in C^\infty(X),$$

which thus is an open continuous map. Consequently, $\mathcal{D}(X)$ is a topological $C^\infty(X)$-algebra.

Given a section σ of $Y^*\to X$ and a function $f\in C^\infty(X)$, let consider the homomorphism (41):

$$(f\sigma)(s) = f\sigma(s) = \sigma(fs), \qquad s\in\mathcal{D}(Y).$$

Since the morphisms $f\sigma$ and σ for any $\sigma\in Y^*(X)$ are continuous and open, the multiplication $s\to fs$ also is an open continuous homomorphism of $\mathcal{D}(Y)$. It follows that $\mathcal{D}(Y)$ is a topological $C^\infty(X)$-module. Accordingly, the homomorphism (41) is a continuous $C^\infty(X)$-homomorphism.

Let $\mathcal{J}^k(\mathcal{D}(Y))$ be a k-order jet module of a $C^\infty(X)$-module $\mathcal{D}(Y)$. It is a submodule of $\mathcal{J}^k(Y(X))$ and, due to the isomorphisms (30) and (40), a submodule

$$\mathcal{J}^k(\mathcal{D}(Y)) = \mathcal{D}(J^k(X)) \subset J^kY(X) \tag{43}$$

of sections with compact support of the jet bundle $J^kY\to X$.

Theorem 11: Let $E\to X$ be a vector bundle. There is one-to-one correspondence between k-order $E(X)$-valued differential operators on sections and sections with compact support of $Y\to X$.

Proof: Of course, any differential operator on $Y(X)$ also is that on $\mathcal{D}(Y)$. Let Δ be a k-order $E(X)$-valued differential operator on a $C^\infty(X)$-module $\mathcal{D}(Y)$. By virtue of Theorem 4 and the isomorphism (43), it defines a unique $C^\infty(X)$-homomorphism

$$\mathfrak{f}^\Delta : \mathcal{J}^k(\mathcal{D}(Y)) = \mathcal{D}(J^k(X)) \to E(X)$$

of sections with compact support of a vector jet bundle $J^kY\to X$ to $E(X)$. Let us show that this morphism is extended to an arbitrary global section s of $J^kY\to X$. A smooth

manifold X admits an atlas whose cover consists of a countable set of open subsets U_i such that that their closures $\overline{U}_i$ are compact [5]. Let $\{f_i\}$ be a subordinate partition of unity, where each f_i is a smooth function with a support $\mathrm{supp} f_i \subset U_i \subset \overline{U}_i$, i.e., with compact support. Each point $x \in X$ has an open neighborhood which intersects only a finite number of $\mathrm{supp} f_i$, and

$$\sum_i f_i(x) = 1, \qquad x \in X.$$

Then one can put

$$s = \sum_i f_i s, \qquad f_i s \in \mathcal{D}(J^k Y) = \mathcal{J}^k(\mathcal{D}(Y)),$$

and define a $C^\infty(X)$-homomorphism

$$\mathfrak{f}^\Delta(s) = \sum_i \mathfrak{f}^\Delta(f_i s)$$

of $J^k Y(X) = \mathcal{J}^k(Y(X))$ to $E(X)$. By virtue of Theorem 4, it provides a k-order $E(X)$-valued differential operator on $Y(X)$.

It follows from Theorem 11 and the isomorphism (32) that

$$\mathrm{Diff}\,_k(\mathcal{D}(Y), E(X)) = \mathrm{Diff}\,_k(Y(X), E(X)) = ((J^k Y)^* \otimes E)(X). \tag{44}$$

Lemma 12: Any $E(X)$-valued differential operator on $\mathcal{D}(Y)$ is $\mathcal{D}(E)$-valued.

Proof: By virtue of Theorem 4, any k-order $E(X)$-valued differential operator Δ on $\mathcal{D}(Y)$ factorizes through a $C^\infty(X)$-homomorphism

$$\mathfrak{f}^\Delta : \mathcal{J}^k(\mathcal{D}(Y)) \to E(X),$$

which is $\mathcal{D}(E)$-valued due to the isomorphism (43).

In particular, let $E = Y$. Then $Y(X)$-valued differential operators on $\mathcal{D}(Y)$ are $\mathcal{D}(Y)$-valued.

Let $Y = R$ and $\mathcal{D}(Y) = \mathcal{D}(X)$ a $C^\infty(X)$-algebra of test functions on X. By virtue of Theorem 11, there is one-to-one correspondence between $E(X)$-valued differential operators on $\mathcal{D}(X)$ and $C^\infty(X)$. Then it follows from the isomorphism (37) that

$$\mathrm{Diff}\,_k(\mathcal{D}(X), E(X)) = ((J^k R)^* \otimes E)(X).$$

In particular, the derivations (6) of $C^\infty(X)$ also are derivations of an $\mathbb{R}$-algebra $\mathcal{D}(X)$. Since $\mathcal{D}(X)$ is an ideal of $C^\infty(X)$, there exists a unique canonical connection $\nabla_u = u$ on $\mathcal{D}(X)$.

Theorem 13: Any $\mathcal{D}(Y)$-valued differential operator on a $C^\infty(X)$-module $\mathcal{D}(Y)$ is continuous.

Proof: Let p_ϕ (39) be a seminorm on $\mathcal{D}(X)$ where $F\phi \in \mathcal{F}_\infty Y$. By virtue of Theorem 4, any k-order $Y(X)$-valued differential operator Δ on $\mathcal{D}(Y)$ yields a bundle morphism

$$\bar{\mathfrak{f}}^\Delta : J^kY \underset{X}{\longrightarrow} Y$$

such that, given a section s of $Y \to X$, we have

$$\Delta(s) = \bar{\mathfrak{f}} \circ J^k s.$$

Let us consider its jet prolongations

$$J^r\bar{\mathfrak{f}}^\Delta : J^{r+k}Y \underset{X}{\longrightarrow} J^rY, \qquad J^\infty\bar{\mathfrak{f}}^\Delta : J^\infty Y \underset{X}{\longrightarrow} J^\infty Y.$$

Then we have a smooth real function

$$\begin{aligned} &\phi_\Delta = \phi \circ J^\infty\bar{\mathfrak{f}}^\Delta : J^\infty Y \underset{X}{\longrightarrow} J^\infty Y \to \mathbb{R}, \\ &\phi_\Delta(z) = (\phi \circ J^{[\phi(\pi^\infty_0(z))]}\bar{\mathfrak{f}}^\Delta)(z), \qquad z \in J^\infty Y, \end{aligned} \tag{45}$$

on $J^\infty Y$ which belongs to $\mathcal{F}_\infty Y$. The function ϕ_Δ (45) yields the seminorm p_{ϕ_Δ} (39) on $\mathcal{D}(Y)$. Then we have

$$p_\phi(\Delta(s)) = p_{\phi_\Delta}(s).$$

It follows that $p_\phi(\Delta(s)) < \varepsilon$ iff $p_{\phi_\Delta}(s) < \varepsilon$ and, consequently, Δ is an open continuous map.

5 Differential operators on Schwartz distributions

Given a LF-space $\mathcal{D}(Y)$ of sections with compact support of a vector bundle $Y \to X$, let $\mathcal{D}(Y)'$ be the topological dual of $\mathcal{D}(Y)$. Its elements are continuous forms

$$\mathcal{D}(Y)' \ni \psi : \mathcal{D}(Y) \ni s \to < s, \psi > \in \mathbb{R}$$

on $\mathcal{D}(Y)$ called the Schwartz distributions. The vector space $\mathcal{D}(Y)'$ is provided with the strong topology (which coincides with all topologies of uniform converges). It is nuclear, and the topological dual of $\mathcal{D}(Y)'$ is $\mathcal{D}(Y)$. A LF-topology of $\mathcal{D}(Y)$ also coincides with all topologies of uniform converges [11].

For instance, let $Y = R$ (34). Then $\mathcal{D}(Y) = \mathcal{D}(X)$ is the space of test functions on a manifold X and its topological dual $\mathcal{D}(X)'$ is the familiar space of Schwartz distributions on test functions on a manifold X. Since a LF-topology is finer than the topology on $\mathcal{D}(X)$ induced by the inductive limit topology of the space $K(X)$ of continuous real functions on X, any measure on X exemplifies a Schwartz distribution. For instance, any density

$$L \in \overset{n}{\wedge} T^*X(X), \qquad n = \dim X,$$

on an oriented manifold X is a Schwartz distribution on $\mathcal{D}(X)$.

Let Δ be a continuous homomorphism of a topological vector space $\mathcal{D}(Y)$. Then

$$\Delta' : \mathcal{D}(Y)' \ni \psi \to \psi \circ \Delta \in \mathcal{D}(Y)', \tag{46}$$

is a morphism of the topological dual $\mathcal{D}(Y)'$ of $\mathcal{D}(Y)$. It is called the transpose or the dual of Δ. We have

$$< \Delta(s), \psi >=< s, \Delta'(\psi) > .$$

Since topologies on $\mathcal{D}(Y)$ and $\mathcal{D}(Y)'$ coincide with the weak ones, the transpose operator Δ' (46) is continuous.

For instance, the transpose (46) of the multiplication $s \to fs$, $f \in C^\infty(X)$, in $\mathcal{D}(Y)$ is the multiplication

$$\psi \to f\psi, \qquad < fs, \psi >=< s, f\psi > \tag{47}$$

which makes $\mathcal{D}(Y)'$ into a topological $C^\infty(X)$-module. In particular, $\mathcal{D}(X)'$ also is a $C^\infty(X)$-module.

Let σ be a global section of the dual bundle $Y^* \to X$. By virtue of Lemma 10, it defines the continuous $C^\infty(X)$-homomorphism (41) of $\mathcal{D}(Y)$ to $\mathcal{D}(X)$ and, accordingly, the dual $C^\infty(X)$-homomorphism

$$\sigma' : \mathcal{D}(X)' \ni \xi \to \xi \circ \sigma \in \mathcal{D}(Y)'. \tag{48}$$

It follows that there is a $C^\infty(X)$-module monomorphism

$$Y^*(X) \underset{C^\infty(X)}{\otimes} \mathcal{D}(X)' \to \mathcal{D}(Y)' \tag{49}$$

such that

$$(\sigma \otimes \xi)(s) = \xi((s, \sigma)), \qquad s \in \mathcal{D}(Y), \qquad \sigma \in Y^*(X), \qquad \xi \in \mathcal{D}(X)'.$$

Theorem 14: The monomorphism (48) is a $C^\infty(X)$-module isomorphism

$$\mathcal{D}(Y)' = Y^*(X) \underset{C^\infty(X)}{\otimes} \mathcal{D}(X)'. \tag{50}$$

Proof: A vector bundle $Y \to X$ of fibre dimension m admits a finite atlas $\{(U_i, h_i), \rho_{ij}\}$, $i, j = 1, \ldots, k$, [5]. Given a smooth partition of unity $\{f_i\}$ subordinate to a cover $\{U_i\}$, let us put

$$l_i = f_i(f_1^2 + \cdots + f_k^2)^{-1/2}.$$

It is readily observed that $\{l_i^2\}$ also is a partition of unity subordinate to $\{U_i\}$. Then any section $s \in \mathcal{D}(Y)$ is represented by a tuple $(s_1, \ldots, s_k)$ of local $\mathbb{R}^m$-valued functions $s_i = h_i \circ s|_{U_i}$ which fulfil the relations

$$s_i = \sum_j \rho_{ij}(s_j) l_j^2. \tag{51}$$

Let us consider a topological vector space

$$\overset{mk}{\oplus}\mathcal{D}(X), \tag{52}$$

which also is a topological $C^\infty(X)$-module. There are both a continuous $C^\infty(X)$-monomorphism

$$\gamma:\mathcal{D}(Y)\ni s\to(l_1s_1,\ldots,l_ks_k)\in\overset{mk}{\oplus}\mathcal{D}(X)$$

and a continuous $C^\infty(X)$-epimorphism

$$\Phi:\overset{mk}{\oplus}\mathcal{D}(X)\ni(t_1,\ldots,t_k)\to(\widetilde{s}_1,\ldots,\widetilde{s}_k)\in\mathcal{D}(Y),\qquad \widetilde{s}_i=\sum_j\rho_{ij}(l_jt_j). \tag{53}$$

In view of the relations (51),

$$\Phi\circ\gamma=\mathrm{Id}\,\mathcal{D}(Y),$$

and we have a decomposition

$$\begin{aligned}&\overset{mk}{\oplus}\mathcal{D}(X)=\gamma(\mathcal{D}(Y))\oplus\mathrm{Ker}\,\Phi,\\ &t_i=[l_i\sum_j\rho_{ij}(l_jt_j)]+[t_i-l_i\sum_j\rho_{ij}(l_jt_j)],\end{aligned} \tag{54}$$

where $\gamma(\mathcal{D}(Y))$ consists of elements (t_i) satisfying the condition

$$t_i=l_i\sum_j\rho_{ij}(l_jt_j). \tag{55}$$

The topological dual of the topological vector space (52) is a $C^\infty(X)$-module

$$\overset{mk}{\oplus}\mathcal{D}(X)' \tag{56}$$

with elements $(\overline{t}_1,\ldots,\overline{t}_k)$. The epimorphism Φ (53) yields a $C^\infty(X)$-monomorphism

$$\Phi':\mathcal{D}(Y)'\ni\xi\to\xi\circ\Phi\in\overset{mk}{\oplus}\mathcal{D}(X)'$$

such that $\Phi'(\mathcal{D}(Y)')$ vanishes on $\mathrm{Ker}\,\Phi$ in the decomposition (54). To describe $\Phi'(\mathcal{D}(Y)')$, let us consider the dual vector bundle $Y^*\to X$ provided with the conjugate atlas $\{(U_i,\overline{h}_i),\overline{\rho}_{ij}\}$ such that, for arbitrary sections s of $Y\to X$ and σ of $Y^*\to X$, the equality

$$(h_i\circ s,\overline{h}_i\circ\sigma)|_{U_i\cap U_j}=(\rho_{ij}\circ h_j\circ s,\overline{\rho}_{ij}\circ\overline{h}_j\circ\overline{s})=(h_j\circ s,\overline{h}_j\circ\sigma)_{U_i\cap U_j} \tag{57}$$

holds. Then it is readily verified that the image $\Phi'(\mathcal{D}(Y)')$ of $\mathcal{D}(Y)'$ in the $C^\infty(X)$-module (56) consists of elements $(\overline{t}_i)$ satisfying the condition

$$\overline{t}_i=l_i\sum_j\overline{\rho}_{ij}(l_jt_j) \tag{58}$$

(cf. the condition (55)). This fact leads to the isomorphism (50).

Since Schwartz distributions on sections with compact support of a vector bundle $Y \to X$ constitute a $C^\infty(X)$ module $\mathcal{D}(Y)'$, differential operators on them can be introduced in accordance with Definition 1.

We restrict our consideration to $\mathcal{D}(Y)'$-valued differential operators on $\mathcal{D}(Y)'$. Of course, any multiplication (47) is a zero-order differential operator on $\mathcal{D}(Y)'$.

In accordance with Theorem 13, any $\mathcal{D}(Y)$-valued differential operator Δ on a $C^\infty(X)$-module $\mathcal{D}(Y)$ of sections with compact support is a continuous morphism of a topological vector space $\mathcal{D}(Y)$. Then it defines the dual morphism Δ' (46) of $\mathcal{D}(Y)'$.

Theorem 15: The transpose Δ' (46) of a k-order differential operator Δ on $\mathcal{D}(Y)$ is a differential operator on $\mathcal{D}(Y)'$ in accordance with Definition 1.

Proof: The proof is based on the fact that $\delta_f\Delta'$, $f \in C^\infty(X)$, (2) is the transpose of $-\delta_f\Delta$.

For instance, any connection ∇ (33) on $Y(X)$ and, consequently, on $\mathcal{D}(Y)$ define the transpose ∇'_u on $\mathcal{D}(Y)'$ for any vector field u on X. We have

$$< \phi, \nabla'_u(f\psi) >=< f\nabla_u(\phi), \psi >=< \nabla(f\phi), \psi > - < \mathbf{L}_u(f)\phi, \psi >=$$
$$< \phi, f\nabla'_u(\psi) > + < \phi, \mathbf{L}_{-u}(f)\psi > .$$

A glance at this equality shows that $-\nabla'_u$ is a connection on $\mathcal{D}(Y)'$.

In particular, let $Y = R$, and let $\mathbf{L}_u$, $u \in TX$, be the derivation (6) of $\mathcal{D}(X)$. Its transpose $\mathbf{L}'_u$ is called the Lie derivative of Schwartz distributions $\psi \in \mathcal{D}(X)'$ along u. In particular, if

$$\mathcal{D}(X)' \ni \psi = \overline{\psi} d^n x$$

is a density on X, then

$$\mathbf{L}'_u(\psi) = \mathbf{L}_{-u}(\psi) = -d(u\rfloor\psi) = -\partial_\mu(u^\mu\overline{\psi})d^n x.$$

It is a derivation of $\mathcal{D}(X)'$ because

$$\mathbf{L}'_u(f\psi) = \mathbf{L}_{-u}(f\psi) = \mathbf{L}_{-u}(f)\psi + f\mathbf{L}_{-u}(\psi).$$

The transpose Δ' (46) on $\mathcal{D}(Y)'$ of a differential operator Δ on $\mathcal{D}(Y)$ is continuous.

However, a differential operator on $\mathcal{D}(Y)'$ need not be the transpose of a differential operator on $\mathcal{D}(Y)$. Since $\mathcal{D}(Y)$ is reflexive and topologies on $\mathcal{D}(Y)$ and $\mathcal{D}(Y)'$ coincide with the weak ones, one can show the following.

Theorem 16: A differential operator on $\mathcal{D}(Y)'$ is the transpose of a differential operator on $\mathcal{D}(Y)$ iff it is continuous.

It follows that there is one-to-one correspondence between continuous differential operators on $\mathcal{D}(Y)'$ and differential operators on $\mathcal{D}(Y)$ whose module is isomorphic to

$$\mathrm{Diff}_k(\mathcal{D}(Y)) = ((J^kY)^* \underset{X}{\otimes} Y)(X) \tag{59}$$

in accordance with the isomorphism (44).

A k-order jet module $\mathcal{J}^k(\mathcal{D}(Y)')$ of a $C^\infty(X)$-module $\mathcal{D}(Y)'$ is introduced in accordance with Definition 3. By virtue of Theorem 4, any k-order differential operator Δ on $\mathcal{D}(Y)'$ is represented by a $C^\infty(X)$-homomorphism

$$\mathfrak{f}^\Delta : \mathcal{J}^k(\mathcal{D}(Y)') \to \mathcal{D}(Y)'.$$

If $\mathfrak{f}^\Delta(\mathcal{D}(Y)) \subset \mathcal{D}(Y)$, then $\Delta|_{\mathcal{D}(Y)}$ is a differential operator on $\mathcal{D}(Y)$ whose transpose is Δ on $\mathcal{D}(Y)'$. Consequently, a differential operator Δ on $\mathcal{D}(Y)'$ is not the transpose of that on $\mathcal{D}(Y)$ iff it does not send $\mathcal{D}(Y) \subset \mathcal{D}(Y)'$ onto itself.

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Differential operators on Schwartz distributions. Jet formalism

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Differential operators on Schwartz distributions conventionally are defined as the transpose of differential operators on functions with compact support. They do not exhaust all differential operators. We follow algebraic formalism of differential operators on modules over commutative rings. In a general setting, Schwartz distributions on sections with compact support of vector bundles over an arbitrary smooth manifold are considered.

1 Introduction

Quantum field theory provides examples of differential operators and differential equations on distributions.

Let U be an open subset of $\mathbb{R}^n$, $\mathcal{D}(U)$ the space of smooth real functions with compact support, and $\mathcal{D}(U)'$ the space of Schwartz distribution on $\mathcal{D}(U)$. Differential operators on $\mathcal{D}(U)'$ conventionally are defined as the transpose of differential operators on $\mathcal{D}(U)$ [2]. However, they do not exhaust all differential operators on $\mathcal{D}(U)'$.

We follow algebraic formalism of differential operators on modules over commutative rings and jets of modules (Section 2) [3, 7].

In a general setting, Schwartz distributions on sections with compact support of a vector bundles Y on an arbitrary smooth manifold X are considered. We follow familiar formalism of distributions, not the nonlinear ones [8].

Let $Y(X)$ denote a $C^\infty(X)$-module of global sections of $Y \to X$. Let $E \to X$ be a vector bundle. The $E(X)$-valued differential operators defined on $Y(X)$ as a $C^\infty(X)$-module coincide with the familiar ones (Section 3). They constitute the $C^\infty(X)$-module (32).

Let $\mathcal{D}(Y)$ be a $C^\infty(X)$-module of sections with compact support of $Y \to X$. It is provided with a LF-topology similar to that on $\mathcal{D}(U)$. With this topology, $\mathcal{D}(U)$ is a nuclear vector space and a topological $C^\infty(X)$-module. Differential operators on $\mathcal{D}(Y)$ as a $C^\infty(X)$-module are defined (Section 4). We show that that there is one-to-one correspondence between $E(X)$-valued differential operators on $\mathcal{D}(Y)$ and those on $Y(X)$ (Theorem 11), and that $\mathcal{D}(Y)$-valued differential operators on $\mathcal{D}(Y)$ are continuous (Theorem 13).

Let $\mathcal{D}(Y)'$ be the topological dual of $\mathcal{D}(Y)$ endowed with the strong topology. It is a nuclear vector space whose topological dual is $\mathcal{D}(Y)$, and it is a topological $C^\infty(X)$-module. Differential operators on $\mathcal{D}(Y)'$ as a $C^\infty(X)$-module are defined. In particular, the transpose of any $\mathcal{D}(Y)$-valued differential operator on $\mathcal{D}(Y)$ is a differential operator on $\mathcal{D}(Y)'$ (Theorem 15). However, a differential operator on $\mathcal{D}(Y')$ need not be of this

type. We show that a $\mathcal{D}(Y)'$-valued differential operator on $\mathcal{D}(Y)'$ is the transpose of a $\mathcal{D}(Y)$-valued differential operator on $\mathcal{D}(Y)$ iff it is continuous (Theorem 16).

2 Differential operators on modules

This Section summarizes the relevant material on differential operator on modules over a commutative ring [3, 7, 10].

Let $\mathcal{K}$ be a commutative ring (i.e., a commutative unital algebra) and $\mathcal{A}$ a commutative $\mathcal{K}$-ring. Let P and Q be $\mathcal{A}$-modules. The $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ of $\mathcal{K}$-homomorphisms $\Phi : P \to Q$ can be endowed with the two different $\mathcal{A}$-module structures

$$(a\Phi)(p) = a\Phi(p), \qquad (a \bullet \Phi)(p) = \Phi(ap), \qquad a \in \mathcal{A}, \quad p \in P. \tag{1}$$

We refer to the second one as a $\mathcal{A}^\bullet$-module structure. Let us put

$$\delta_a\Phi = a\Phi - a \bullet \Phi, \qquad a \in \mathcal{A}. \tag{2}$$

Definition 1: An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is called a k-order Q-valued differential operator on P if

$$\delta_{a_0} \circ \cdots \circ \delta_{a_k}\Delta = 0$$

for any tuple of $k+1$ elements $a_0,\ldots,a_k$ of $\mathcal{A}$. The set $\mathrm{Diff}_k(P,Q)$ of these operators inherits the $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures (1).

In particular, zero order differential operators are $\mathcal{A}$-homomorphisms $P \to Q$. A first order differential operator Δ satisfies the condition

$$\delta_b \circ \delta_a \Delta(p) = ba\Delta(p) - b\Delta(ap) - a\Delta(bp) + \Delta(abp) = 0, \quad a, b \in \mathcal{A}.$$

Let $P = \mathcal{A}$. Any zero order Q-valued differential operator Δ on $\mathcal{A}$ is defined by its value $\Delta(\mathbf{1})$. Then there is an isomorphism

$$\mathrm{Diff}_0(\mathcal{A},Q) = Q$$

via the association

$$Q \ni q \to \Delta_q \in \mathrm{Diff}_0(\mathcal{A},Q), \qquad \Delta_q(\mathbf{1}) = q,$$

A first order Q-valued differential operator Δ on $\mathcal{A}$ fulfils the condition

$$\Delta(ab) = b\Delta(a) + a\Delta(b) - ba\Delta(\mathbf{1}), \qquad a, b \in \mathcal{A}.$$

It is a Q-valued derivation of $\mathcal{A}$ if $\Delta(\mathbf{1}) = 0$, i.e., the Leibniz rule

$$\Delta(ab) = \Delta(a)b + a\Delta(b), \qquad a, b \in \mathcal{A}, \tag{3}$$

holds. Any first order differential operator on $\mathcal{A}$ falls into the sum

$$\Delta(a) = a\Delta(\mathbf{1}) + [\Delta(a) - a\Delta(\mathbf{1})]$$

of a zero order differential operator and a derivation. Accordingly, there is an $\mathcal{A}$-module decomposition

$$\mathrm{Diff}\,_1(\mathcal{A}, Q) = Q \oplus \mathfrak{d}(\mathcal{A}, Q), \tag{4}$$

where $\mathfrak{d}(\mathcal{A}, Q)$ is an $\mathcal{A}$-module of Q-valued derivations of $\mathcal{A}$.

If $P = Q = \mathcal{A}$, the derivation module $\mathfrak{d}\mathcal{A}$ of $\mathcal{A}$ is a Lie $\mathcal{K}$-algebra. Accordingly, the decomposition (4) takes the form

$$\mathrm{Diff}\,_1(\mathcal{A}) = \mathcal{A} \oplus \mathfrak{d}\mathcal{A}. \tag{5}$$

Example 1: Let X be an n-dimensional real smooth manifold coordinated by x^μ, and let $C^\infty(X)$ be an $\mathbb{R}$-ring of smooth real functions on X. There is one-to-one correspondence between the derivations of $C^\infty(X)$ and the vector fields on X. It is given by the expression

$$T(X) = TX(X) \ni u \leftrightarrow \mathbf{L}_u \in \mathfrak{d}C^\infty(X), \qquad \mathbf{L}_u(f) = u^\mu \partial_\mu f, \qquad f \in C^\infty(X), \tag{6}$$

where $\mathbf{L}_u$ denotes the Lie derivative along u.

The study of Q-valued differential operators on an $\mathcal{A}$-module P is reduced to that of Q-valued differential operators on a ring $\mathcal{A}$ as follows.

Theorem 2: Let us consider an $\mathcal{A}$-homomorphism

$$h_k : \mathrm{Diff}\,_k(\mathcal{A}, Q) \to Q, \qquad h_k(\Delta) = \Delta(\mathbf{1}). \tag{7}$$

Any k-order Q-valued differential operator $\Delta \in \mathrm{Diff}\,_k(P, Q)$ on P uniquely factorizes as

$$\Delta : P \xrightarrow{\mathfrak{f}_\Delta} \mathrm{Diff}\,_k(\mathcal{A}, Q) \xrightarrow{h_k} Q \tag{8}$$

through the homomorphism h_k (7) and some homomorphism

$$\mathfrak{f}_\Delta : P \to \mathrm{Diff}\,_k(\mathcal{A}, Q), \qquad (\mathfrak{f}_\Delta p)(a) = \Delta(ap), \qquad a \in \mathcal{A}, \tag{9}$$

of an $\mathcal{A}$-module P to an $\mathcal{A}^\bullet$-module $\mathrm{Diff}\,_k(\mathcal{A}, Q)$. The assignment $\Delta \to \mathfrak{f}_\Delta$ defines an $\mathcal{A}^\bullet - \mathcal{A}$-module isomorphism

$$\mathrm{Diff}\,_k(P, Q) = \mathrm{Hom}\,_{\mathcal{A}-\mathcal{A}^\bullet}(P, \mathrm{Diff}\,_k(\mathcal{A}, Q)). \tag{10}$$

In a different way, k-order differential operators on a module P are represented by zero order differential operators on a module of k-order jets of P as follows.

Given an $\mathcal{A}$-module P, let us consider a tensor product $\mathcal{A} \otimes_\mathcal{K} P$ of $\mathcal{K}$-modules $\mathcal{A}$ and P. We put

$$\delta^b(a \otimes p) = (ba) \otimes p - a \otimes (bp), \qquad p \in P, \qquad a, b \in \mathcal{A}. \tag{11}$$

Let us denote by μ^{k+1} a submodule of $\mathcal{A}\otimes_{\mathcal{K}} P$ generated by elements of the type

$$\delta^{b_0}\circ\cdots\circ\delta^{b_k}(a\otimes p).$$

Definition 3: A k-order jet module $\mathcal{J}^k(P)$ of a module P is the quotient of the $\mathcal{K}$-module $\mathcal{A}\otimes_{\mathcal{K}} P$ by μ^{k+1}. We denote its elements $c\otimes_k p$.

In particular, a first order jet module $\mathcal{J}^1(P)$ is generated by elements $\mathbf{1}\otimes_1 p$ modulo the relations

$$\delta^a\circ\delta^b(\mathbf{1}\otimes_1 p) = ab\otimes_1 p - b\otimes_1 (ap) - a\otimes_1 (bp) + \mathbf{1}\otimes_1 (abp) = 0. \tag{12}$$

A $\mathcal{K}$-module $\mathcal{J}^k(P)$ is endowed with the $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures

$$b(a\otimes_k p) = ba\otimes_k p, \qquad b\bullet(a\otimes_k p) = a\otimes_k (bp). \tag{13}$$

There exists a homomorphism

$$J^k : P\ni p\to \mathbf{1}\otimes_k p\in\mathcal{J}^k(P) \tag{14}$$

of an $\mathcal{A}$-module P to an $\mathcal{A}^\bullet$-module $\mathcal{J}^k(P)$ such that $\mathcal{J}^k(P)$, seen as an $\mathcal{A}$-module, is generated by elements $J^k p$, $p\in P$.

Due to the natural monomorphisms $\mu^r\to\mu^k$ for all $r>k$, there are $\mathcal{A}$-module epimorphisms of jet modules

$$\pi^{i+1}_i : \mathcal{J}^{i+1}(P)\to\mathcal{J}^i(P).$$

In particular,

$$\pi^1_0 : \mathcal{J}^1(P)\ni a\otimes_1 p\to ap\in P. \tag{15}$$

Theorem 4: Any k-order Q-valued differential operator Δ on an $\mathcal{A}$-module P factorizes uniquely

$$\Delta : P\xrightarrow{J^k}\mathcal{J}^k(P)\xrightarrow{\mathfrak{f}^\Delta} Q$$

through the homomorphism J^k (14) and some $\mathcal{A}$-homomorphism $\mathfrak{f}^\Delta : \mathcal{J}^k(P)\to Q$. The association $\Delta\to\mathfrak{f}^\Delta$ yields an $(\mathcal{A}^\bullet-\mathcal{A})$-module isomorphism

$$\mathrm{Diff}_k(P,Q) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{J}^k(P),Q). \tag{16}$$

Let us consider jet modules $\mathcal{J}^k=\mathcal{J}^k(\mathcal{A})$ of a ring $\mathcal{A}$ itself. In particular, the first order jet module $\mathcal{J}^1$ consists of the elements $a\otimes_1 b$, $a,b\in\mathcal{A}$, subject to the relations

$$ab\otimes_1\mathbf{1} - b\otimes_1 a - a\otimes_1 b + \mathbf{1}\otimes_1 (ab) = 0. \tag{17}$$

The $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures (13) on $\mathcal{J}^1$ read

$$c(a\otimes_1 b)=(ca)\otimes_1 b, \qquad c\bullet(a\otimes_k b)=a\otimes_1(cb)=(a\otimes_1 b)c.$$

Theorems 2 and 4 are completed with forthcoming Theorem 5 so that any one of them is a corollary of the others.

Theorem 5: There is an isomorphism

$$\mathcal{J}^k(P)=\mathcal{J}^k\underset{\mathcal{A}^\bullet-\mathcal{A}}{\otimes}P, \qquad (a\otimes_k bp)\leftrightarrow(a\otimes_1 b)\otimes p. \tag{18}$$

Then we have the $(\mathcal{A}-\mathcal{A}^\bullet)$-module isomorphisms

$$\begin{aligned}&\mathrm{Hom}\,_{\mathcal{A}}(\mathcal{J}^k\otimes P,Q)=\mathrm{Hom}\,_{\mathcal{A}-\mathcal{A}^\bullet}(P,\mathrm{Hom}\,_{\mathcal{A}}(\mathcal{J}^k,Q))=\\ &\qquad\mathrm{Hom}\,_{\mathcal{A}-\mathcal{A}^\bullet}(P,\mathrm{Diff}\,_k(\mathcal{A},Q))=\mathrm{Diff}\,_k(P,Q)=\mathrm{Hom}\,_{\mathcal{A}}(\mathcal{J}^k(P),Q).\end{aligned}$$

Besides the monomorphism (14):

$$J^1:\mathcal{A}\ni a\to\mathbf{1}\otimes_1 a\in\mathcal{J}^1,$$

there exists an $\mathcal{A}$-module monomorphism

$$i_1:\mathcal{A}\ni a\to a\otimes_1\mathbf{1}\in\mathcal{J}^1.$$

With these monomorphisms, we have the canonical $\mathcal{A}$-module splitting

$$\mathcal{J}^1=i_1(\mathcal{A})\oplus\mathcal{O}^1, \qquad J^1(b)=\mathbf{1}\otimes_1 b=b\otimes_1\mathbf{1}+(\mathbf{1}\otimes_1 b-b\otimes_1\mathbf{1}), \tag{19}$$

where the $\mathcal{A}$-module $\mathcal{O}^1$ is generated by elements $\mathbf{1}\otimes_1 b-b\otimes_1\mathbf{1}$ for all $b\in\mathcal{A}$. Let us consider a $\mathcal{K}$-homomorphism

$$d^1:\mathcal{A}\ni b\to\mathbf{1}\otimes_1 b-b\otimes_1\mathbf{1}\in\mathcal{O}^1. \tag{20}$$

This is a $\mathcal{O}^1$-valued derivation of a $\mathcal{K}$-ring $\mathcal{A}$ which obeys the Leibniz rule

$$d^1(ab)=\mathbf{1}\otimes_1 ab-ab\otimes_1\mathbf{1}+a\otimes_1 b-a\otimes_1 b=ad^1b+(d^1a)b.$$

It follows from the relation (17) that $ad^1b=(d^1b)a$ for all $a,b\in\mathcal{A}$. Thus, seen as an $\mathcal{A}$-module, $\mathcal{O}^1$ is generated by elements d^1a for all $a\in\mathcal{A}$.

Let $\mathcal{O}^{1*}=\mathrm{Hom}\,_{\mathcal{A}}(\mathcal{O}^1,\mathcal{A})$ be the dual of an $\mathcal{A}$-module $\mathcal{O}^1$. In view of the splittings (5) and (19), the isomorphism (16) leads to the duality relation

$$\mathfrak{d}\mathcal{A}=\mathcal{O}^{1*}, \qquad \mathfrak{d}\mathcal{A}\ni u\leftrightarrow\phi_u\in\mathcal{O}^{1*}, \qquad \phi_u(d^1a)=u(a), \qquad a\in\mathcal{A}. \tag{21}$$

Example 2: If $\mathcal{A} = C^\infty(X)$ in Example 1, then $\mathcal{O}^1 = \mathcal{O}^1(X)$ is a module of differential one-forms on X, and there is an isomorphism $\mathcal{O}^1(X) = T(X)^*$, besides the isomorphism $T(X) = \mathcal{O}^1(X)^*$ (21).

Let us return to the first order jet module $\mathcal{J}^1(P)$ of an $\mathcal{A}$-module P. Due to the isomorphism (18), the isomorphism (19) leads to the splitting

$$\begin{aligned} &\mathcal{J}^1(P) = (\mathcal{A} \oplus \mathcal{O}^1) \underset{\mathcal{A}^\bullet - \mathcal{A}}{\otimes} P = (\mathcal{A} \underset{\mathcal{A}}{\otimes} P) \oplus (\mathcal{O}^1 \underset{\mathcal{A}^\bullet - \mathcal{A}}{\otimes} P), \\ &a \otimes_1 bp \leftrightarrow (ab + ad^1(b)) \otimes p. \end{aligned} \tag{22}$$

Applying the epimorphism π^1_0 (15) to this splitting, one obtains the short exact sequence of $(\mathcal{A} - \mathcal{A}^\bullet)$-modules

$$\begin{aligned} &0 \longrightarrow \mathcal{O}^1 \underset{\mathcal{A}^\bullet - \mathcal{A}}{\otimes} P \to \mathcal{J}^1(P) \xrightarrow{\pi^1_0} P \longrightarrow 0, \\ &(a \otimes_1 b - ab \otimes_1 \mathbf{1}) \otimes p \to (c \otimes_1 \mathbf{1} + a \otimes_1 b - ab \otimes_1 \mathbf{1}) \otimes p \to cp. \end{aligned} \tag{23}$$

It is canonically split by the $\mathcal{A}^\bullet$-homomorphism

$$P \ni ap \to \mathbf{1} \otimes_1 ap = a \otimes_1 p + d^1(a) \otimes_1 p \in \mathcal{J}^1(P).$$

However, it need not be split by an $\mathcal{A}$-homomorphism, unless P is a projective $\mathcal{A}$-module.

Definition 6: A connection on an $\mathcal{A}$-module P is defined as an $\mathcal{A}$-homomorphism

$$\Gamma : P \to \mathcal{J}^1(P), \qquad \Gamma(ap) = a\Gamma(p), \tag{24}$$

which splits the exact sequence (23).

Given the splitting Γ (24), let us define a complementary morphism

$$\nabla = J^1 - \Gamma : P \to \mathcal{O}^1 \underset{\mathcal{A}^\bullet - \mathcal{A}}{\otimes} P, \qquad \nabla(p) = \mathbf{1} \otimes_1 p - \Gamma(p). \tag{25}$$

This also is called a connection though it in fact is a covariant differential on a module P. This morphism satisfies the Leibniz rule

$$\nabla(ap) = d^1 a \otimes p + a\nabla(p), \tag{26}$$

i.e., ∇ is first order $(\mathcal{O}^1 \otimes P)$-valued differential operator on P. Thus, we come to the equivalent definition of a connection [6].

Definition 7: A connection on an $\mathcal{A}$-module P is a $\mathcal{K}$-homomorphism ∇ (25) which obeys the Leibniz rule (26).

In view of the isomorphism (21), any connection in Definition 7 determines a connection in the following sense.

Definition 8: A connection on an $\mathcal{A}$-module P is an $\mathcal{A}$-homomorphism

$$\nabla : \mathfrak{d}\mathcal{A} \ni u \to \nabla_u \in \mathrm{Diff}_1(P, P) \tag{27}$$

such that, for each $u \in \mathfrak{d}\mathcal{A}$, the first order differential operator ∇_u obeys the Leibniz rule

$$\nabla_u(ap) = u(a)p + a\nabla_u(p), \quad a \in \mathcal{A}, \quad p \in P. \tag{28}$$

Definitions 7 and 8 are equivalent if $\mathcal{O}^1 = \mathfrak{d}\mathcal{A}^*$. For instance, this is the case of $\mathcal{A} = C^\infty(X)$ in Example 2.

In particular, let P be a commutative $\mathcal{A}$-algebra and $\mathfrak{d}P$ the derivation module of P as a $\mathcal{K}$-algebra. The $\mathfrak{d}P$ is both a P- and $\mathcal{A}$-modules. Then Definition 8 is modified as follows.

Definition 9: A connection on an $\mathcal{A}$-algebra P is an $\mathcal{A}$-homomorphism

$$\nabla : \mathfrak{d}\mathcal{A} \ni u \to \nabla_u \in \mathfrak{d}P \subset \mathrm{Diff}_1(P,P), \tag{29}$$

which is a connection on P as an $\mathcal{A}$-module, i.e., it obeys the Leinbniz rule (28).

For instance, if P is an ideal of $\mathcal{A}$, there is a unique canonical connection $u \to \nabla_u = u$ on P.

3 Differential operators on sections of a vector bundle

Let X be a smooth manifold which is customarily assumed to be Hausdorff and second-countable (i.e., it has a countable base for topology). Consequently, it has a locally compact space which is a union of a countable number of compact subsets, a separable space, a paracompact and completely regular space. Let X be connected and oriented.

Let $Y \to X$ be a vector bundle over X. Its global sections s constitute a $C^\infty(X)$-module $Y(X)$.

Let J^kY be a k-order jet manifold of Y whose elements are k-order jets of sections s of $Y \to X$. It is a vector bundle $J^kY \to X$ over X. There is a $C^\infty(X)$-module isomorphism

$$\mathcal{J}^k(Y(X)) = J^kY(X) \tag{30}$$

of a k-order jet module $\mathcal{J}^k(Y(X))$ of $Y(X)$ and a module $J^kY(X)$ of global sections of a k-order jet bundle $J^kY \to X$ of $Y \to X$ [4, 9].

Let $E \to X$ be a vector bundle and $E(X)$ a $C^\infty(X)$-module of global sections of E. By virtue of Theorem 4, there is the $C^\infty(X)$-module isomorphism (16):

$$\mathrm{Diff}_k(Y(X), E(X)) = \mathrm{Hom}_{C^\infty(X)}(\mathcal{J}^k(Y(X)), E(X)) \tag{31}$$

of the module $\mathrm{Diff}_k(Y(X), E(X))$ of k-order $E(X)$-valued differential operators on $Y(X)$ and the module $\mathrm{Hom}_{C^\infty(X)}(\mathcal{J}^k(Y(X)), E(X))$ of $C^\infty(X)$-homomorphisms of $\mathcal{J}^k(Y(X))$ to $E(X)$. Since $\mathcal{J}^k(Y(X))$ (30) is a projective $C^\infty(X)$-module of finite rank, the isomorphism (31) takes the form

$$\begin{aligned}\mathrm{Diff}_k(Y(X), E(X)) = \mathrm{Hom}_{C^\infty(X)}(\mathcal{J}^k(Y(X)), E(X)) = \\ \mathcal{J}^k(Y(X))^* \underset{C^\infty(X)}{\otimes} E(X) = ((J^kY)^* \otimes E)(X).\end{aligned} \tag{32}$$

It follows that there is one-to-one correspondence between the k-order $E(X)$-valued differential operators on $Y(X)$ and the global sections of the vector bundle $(J^kY)^*\otimes E$ where $(J^kY)^*$ is the dual of a vector bundle $J^kY\to X$.

In particular, let $E=Y$. In accordance with Definition 8, a connection ∇ on a module $Y(X)$ is a $C^\infty(X)$-homomorphism

$$\nabla : \mathfrak{d}C^\infty(X)=T(X)\ni u\to \nabla_u\in \mathrm{Diff}\,_1(Y(X),Y(X)) \tag{33}$$

such that, for each vector field u on X, a first order differential operator ∇_u obeys the Leibniz rule (28):

$$\nabla_u(fs)=(\mathbf{L}_uf)s+f\nabla_us, \qquad s\in Y(X), \qquad f\in C^\infty(X).$$

There is one-to-one correspondence between the connections ∇^Γ on a module $Y(X)$ and the linear connections Γ on a vector bundle $Y\to X$ such that ∇^Γ is the covariant differential with respect to Γ [4, 9].

For instance, let

$$R=X\times\mathbb{R}\to X \tag{34}$$

be a trivial bundle. Its global sections are smooth real functions on X, i.e., $C^\infty(X)=R(X)$. A k-order jet manifold of this bundle is diffeomorphic to a Whitney sum

$$J^kR=R\oplus T^*X\oplus\cdots\oplus\overset{k}{\vee}T^*X \tag{35}$$

of symmetric products of the cotangent bundle T^*X to X. By virtue of the isomorphism (30), a k-order jet module $\mathcal{J}^k(C^\infty(X))=\mathcal{J}^k(R(X))$ of a ring $C^\infty(X)$ is a module of sections $J^kR(X)$ of the vector bundle $J^kR\to X$ (35). Let $E\to X$ be a vector bundle. Then the module $\mathrm{Diff}\,_k(C^\infty(X),E(X))$ of $E(X)$-valued differential operators on $C^\infty(X)$ is isomorphic to a module of sections of the vector bundle $(J^kR)^*\otimes E$ where

$$(J^kR)^*=R\oplus TX\oplus\cdots\oplus\overset{k}{\vee}TX \tag{36}$$

is the dual of the vector bundle (35). Thus, we have

$$\mathrm{Diff}\,_k(C^\infty(X),E(X))=((J^kR)^*\otimes E)(X). \tag{37}$$

In particular, let $E=R$. Then the module of k-order $C^\infty(X)$-valued differential operators on $C^\infty(X)$ is

$$\mathrm{Diff}\,_k(C^\infty(X)=(J^kR)^*(X). \tag{38}$$

For instance, the module $\mathrm{Diff}\,_1(C^\infty(X))$ of first order differential operators on $C^\infty(X)$ is isomorphic to $C^\infty(X)\oplus T(X)$ in accordance with the decomposition (5).

4 Differential operators on sections with compact support

Let us consider a $C^\infty(X)$-module $\mathcal{D}(Y) \subset Y(X)$ of sections with compact support of a vector bundle $Y \to X$. It is endowed with the following topology [1].

Let $J^\infty Y$ be the topological inductive limit of $J^k Y$, $k \in \mathbb{N}$ which is a Fréchet (not smooth) manifold [4, 9]. It is a topological vector bundle

$$\pi^\infty_0 : J^\infty Y \to X.$$

There is a certain class $\mathcal{Q}^0_\infty Y$ of real functions on $J^\infty Y$ called the smooth functions on $J^\infty Y$. Given a function $f \in \mathcal{Q}^0_\infty Y$ and a point $z \in J^\infty Y$, there exists an open neighborhood U of z such that $f|_U$ is the pull-back of a smooth function on some finite order jet manifold $J^k Y$.

Let $\mathcal{F}_\infty Y \subset \mathcal{Q}^0_\infty Y$ denote a subset of smooth functions ϕ on $J^\infty Y$ which are of finite jet order $[\phi(K)]$ on a subset $(\pi^\infty_0)^{-1}(K) \subset J^\infty Y$ over any compact subset $K \subset X$, and which are linear on fibres of $J^\infty Y \to X$. With $\phi \in \mathcal{F}_\infty Y$, one can define a seminorm

$$p_\phi(s) = \sup_{x\in X} |J^\infty s^* \phi| \tag{39}$$

on $\mathcal{D}(Y)$ where $J^\infty s$ denotes the jet prolongation of a section s to a section of $J^\infty Y \to X$. The seminorm (39) is well defined because

$$(J^\infty s^* \phi) = (J^{[\phi(\mathrm{supp}\, s)]} s^* \phi)(x)$$

is a smooth function with compact support on X.

The set of seminorms $p_\phi(s)$ (39) for all functions $\phi \in \mathcal{F}_\infty Y$ on $J^\infty Y$ yields a locally convex topology on $\mathcal{D}(Y)$ called the LF-topology. With this topology, $\mathcal{D}(Y)$ is a nuclear complete reflexive vector space, an inductive limit of a countable family of separable Fréchet spaces [11].

If $Y = R$, the $\mathcal{D}(R) = \mathcal{D}(X)$ is the well-known nuclear space of test functions on a manifold X. There is a $C^\infty(X)$-module isomorphism

$$\mathcal{D}(Y) = Y(X) \underset{C^\infty(X)}{\otimes} \mathcal{D}(X) \subset Y(X) \underset{C^\infty(X)}{\otimes} Y(X) = Y(X). \tag{40}$$

Lemma 10: Any global section σ of the dual vector bundle $Y^* \to X$ yields a continuous homomorphism of the topological vector spaces

$$\sigma : \mathcal{D}(Y) \ni s \to (s, \sigma) \in \mathcal{D}(X). \tag{41}$$

Proof: Let p_F (39) be a seminorm on $\mathcal{D}(X)$ where $F \in \mathcal{F}_\infty R$. A global section σ of $Y^* \to X$ defines a bundle morphism $\sigma : Y \to R$ over X possessing a jet prolongation

$$J^\infty \sigma : J^\infty Y \underset{X}{\longrightarrow} J^\infty R.$$

Then we have a smooth real function

$$
\begin{aligned}
& F_\sigma = F \circ J^\infty \sigma : J^\infty Y \underset{X}{\longrightarrow} J^\infty R \to \mathbb{R}, \\
& F_\sigma(z) = (F \circ J^{[F(\pi_0^\infty(z))]}\sigma)(z), \qquad z \in J^\infty Y,
\end{aligned} \tag{42}
$$

on $J^\infty Y$ which belongs to $\mathcal{F}_\infty Y$. The function F_σ (42) yields the seminorm p_{F_σ} (39) on $\mathcal{D}(Y)$ such that

$$
p_F((s,\sigma)) = p_{F_\sigma}(s)
$$

for all $s \in \mathcal{D}(Y)$. It follows that $p_F((s,\sigma)) < \varepsilon$ iff $p_{F_\sigma}(s) < \varepsilon$ and, consequently, σ (41) is an open continuous map.

In the case of $Y = R$, the homomorphism (41) is a multiplication

$$
g : \mathcal{D}(X) \ni f \to gf \in \mathcal{D}(X), \qquad g \in C^\infty(X),
$$

which thus is an open continuous map. Consequently, $\mathcal{D}(X)$ is a topological $C^\infty(X)$-algebra.

Given a section σ of $Y^* \to X$ and a function $f \in C^\infty(X)$, let consider the homomorphism (41):

$$
(f\sigma)(s) = f\sigma(s) = \sigma(fs), \qquad s \in \mathcal{D}(Y).
$$

Since the morphisms $f\sigma$ and σ for any $\sigma \in Y^*(X)$ are continuous and open, the multiplication $s \to fs$ also is an open continuous homomorphism of $\mathcal{D}(Y)$. It follows that $\mathcal{D}(Y)$ is a topological $C^\infty(X)$-module. Accordingly, the homomorphism (41) is a continuous $C^\infty(X)$-homomorphism.

Let $\mathcal{J}^k(\mathcal{D}(Y))$ be a k-order jet module of a $C^\infty(X)$-module $\mathcal{D}(Y)$. It is a submodule of $\mathcal{J}^k(Y(X))$ and, due to the isomorphisms (30) and (40), a submodule

$$
\mathcal{J}^k(\mathcal{D}(Y)) = \mathcal{D}(J^k(X)) \subset J^k Y(X) \tag{43}
$$

of sections with compact support of the jet bundle $J^k Y \to X$.

Theorem 11: Let $E \to X$ be a vector bundle. There is one-to-one correspondence between k-order $E(X)$-valued differential operators on sections and sections with compact support of $Y \to X$.

Proof: Of course, any differential operator on $Y(X)$ also is that on $\mathcal{D}(Y)$. Let Δ be a k-order $E(X)$-valued differential operator on a $C^\infty(X)$-module $\mathcal{D}(Y)$. By virtue of Theorem 4 and the isomorphism (43), it defines a unique $C^\infty(X)$-homomorphism

$$
\mathfrak{f}^\Delta : \mathcal{J}^k(\mathcal{D}(Y)) = \mathcal{D}(J^k(X)) \to E(X)
$$

of sections with compact support of a vector jet bundle $J^k Y \to X$ to $E(X)$. Let us show that this morphism is extended to an arbitrary global section s of $J^k Y \to X$. A smooth

manifold X admits an atlas whose cover consists of a countable set of open subsets U_i such that that their closures $\overline{U}_i$ are compact [5]. Let $\{f_i\}$ be a subordinate partition of unity, where each f_i is a smooth function with a support $\mathrm{supp} f_i \subset U_i \subset \overline{U}_i$, i.e., with compact support. Each point $x \in X$ has an open neighborhood which intersects only a finite number of $\mathrm{supp} f_i$, and

$$\sum_i f_i(x) = 1, \qquad x \in X.$$

Then one can put

$$s = \sum_i f_i s, \qquad f_i s \in \mathcal{D}(J^k Y) = \mathcal{J}^k(\mathcal{D}(Y)),$$

and define a $C^\infty(X)$-homomorphism

$$\mathfrak{f}^\Delta(s) = \sum_i \mathfrak{f}^\Delta(f_i s)$$

of $J^k Y(X) = \mathcal{J}^k(Y(X))$ to $E(X)$. By virtue of Theorem 4, it provides a k-order $E(X)$-valued differential operator on $Y(X)$.

It follows from Theorem 11 and the isomorphism (32) that

$$\mathrm{Diff}_k(\mathcal{D}(Y), E(X)) = \mathrm{Diff}_k(Y(X), E(X)) = ((J^k Y)^* \otimes E)(X). \tag{44}$$

Lemma 12: Any $E(X)$-valued differential operator on $\mathcal{D}(Y)$ is $\mathcal{D}(E)$-valued.

Proof: By virtue of Theorem 4, any k-order $E(X)$-valued differential operator Δ on $\mathcal{D}(Y)$ factorizes through a $C^\infty(X)$-homomorphism

$$\mathfrak{f}^\Delta : \mathcal{J}^k(\mathcal{D}(Y)) \to E(X),$$

which is $\mathcal{D}(E)$-valued due to the isomorphism (43).

In particular, let $E = Y$. Then $Y(X)$-valued differential operators on $\mathcal{D}(Y)$ are $\mathcal{D}(Y)$-valued.

Let $Y = R$ and $\mathcal{D}(Y) = \mathcal{D}(X)$ a $C^\infty(X)$-algebra of test functions on X. By virtue of Theorem 11, there is one-to-one correspondence between $E(X)$-valued differential operators on $\mathcal{D}(X)$ and $C^\infty(X)$. Then it follows from the isomorphism (37) that

$$\mathrm{Diff}_k(\mathcal{D}(X), E(X)) = ((J^k R)^* \otimes E)(X).$$

In particular, the derivations (6) of $C^\infty(X)$ also are derivations of an $\mathbb{R}$-algebra $\mathcal{D}(X)$. Since $\mathcal{D}(X)$ is an ideal of $C^\infty(X)$, there exists a unique canonical connection $\nabla_u = u$ on $\mathcal{D}(X)$.

Theorem 13: Any $\mathcal{D}(Y)$-valued differential operator on a $C^\infty(X)$-module $\mathcal{D}(Y)$ is continuous.

Proof: Let p_ϕ (39) be a seminorm on $\mathcal{D}(X)$ where $F\phi \in \mathcal{F}_\infty Y$. By virtue of Theorem 4, any k-order $Y(X)$-valued differential operator Δ on $\mathcal{D}(Y)$ yields a bundle morphism

$$\overline{\mathfrak{f}}^\Delta : J^k Y \underset{X}{\longrightarrow} Y$$

such that, given a section s of $Y \to X$, we have

$$\Delta(s) = \overline{\mathfrak{f}} \circ J^k s.$$

Let us consider its jet prolongations

$$J^r \overline{\mathfrak{f}}^\Delta : J^{r+k} Y \underset{X}{\longrightarrow} J^r Y, \qquad J^\infty \overline{\mathfrak{f}}^\Delta : J^\infty Y \underset{X}{\longrightarrow} J^\infty Y.$$

Then we have a smooth real function

$$\begin{aligned} &\phi_\Delta = \phi \circ J^\infty \overline{\mathfrak{f}}^\Delta : J^\infty Y \underset{X}{\longrightarrow} J^\infty Y \to \mathbb{R}, \\ &\phi_\Delta(z) = (\phi \circ J^{[\phi(\pi^\infty_0(z))]} \overline{\mathfrak{f}}^\Delta)(z), \qquad z \in J^\infty Y, \end{aligned} \tag{45}$$

on $J^\infty Y$ which belongs to $\mathcal{F}_\infty Y$. The function ϕ_Δ (45) yields the seminorm p_{ϕ_Δ} (39) on $\mathcal{D}(Y)$. Then we have

$$p_\phi(\Delta(s)) = p_{\phi_\Delta}(s).$$

It follows that $p_\phi(\Delta(s)) < \varepsilon$ iff $p_{\phi_\Delta}(s) < \varepsilon$ and, consequently, Δ is an open continuous map.

5 Differential operators on Schwartz distributions

Given a LF-space $\mathcal{D}(Y)$ of sections with compact support of a vector bundle $Y \to X$, let $\mathcal{D}(Y)'$ be the topological dual of $\mathcal{D}(Y)$. Its elements are continuous forms

$$\mathcal{D}(Y)' \ni \psi : \mathcal{D}(Y) \ni s \to < s, \psi > \in \mathbb{R}$$

on $\mathcal{D}(Y)$ called the Schwartz distributions. The vector space $\mathcal{D}(Y)'$ is provided with the strong topology (which coincides with all topologies of uniform converges). It is nuclear, and the topological dual of $\mathcal{D}(Y)'$ is $\mathcal{D}(Y)$. A LF-topology of $\mathcal{D}(Y)$ also coincides with all topologies of uniform converges [11].

For instance, let $Y = R$ (34). Then $\mathcal{D}(Y) = \mathcal{D}(X)$ is the space of test functions on a manifold X and its topological dual $\mathcal{D}(X)'$ is the familiar space of Schwartz distributions on test functions on a manifold X. Since a LF-topology is finer than the topology on $\mathcal{D}(X)$ induced by the inductive limit topology of the space $K(X)$ of continuous real functions on X, any measure on X exemplifies a Schwartz distribution. For instance, any density

$$L \in \overset{n}{\wedge} T^* X(X), \qquad n = \dim X,$$

on an oriented manifold X is a Schwartz distribution on $\mathcal{D}(X)$.

Let Δ be a continuous homomorphism of a topological vector space $\mathcal{D}(Y)$. Then

$$\Delta' : \mathcal{D}(Y)' \ni \psi \to \psi \circ \Delta \in \mathcal{D}(Y)', \tag{46}$$

is a morphism of the topological dual $\mathcal{D}(Y)'$ of $\mathcal{D}(Y)$. It is called the transpose or the dual of Δ. We have

$$< \Delta(s), \psi >=< s, \Delta'(\psi) > .$$

Since topologies on $\mathcal{D}(Y)$ and $\mathcal{D}(Y)'$ coincide with the weak ones, the transpose operator Δ' (46) is continuous.

For instance, the transpose (46) of the multiplication $s \to fs$, $f \in C^\infty(X)$, in $\mathcal{D}(Y)$ is the multiplication

$$\psi \to f\psi, \qquad < fs, \psi >=< s, f\psi > \tag{47}$$

which makes $\mathcal{D}(Y)'$ into a topological $C^\infty(X)$-module. In particular, $\mathcal{D}(X)'$ also is a $C^\infty(X)$-module.

Let σ be a global section of the dual bundle $Y^* \to X$. By virtue of Lemma 10, it defines the continuous $C^\infty(X)$-homomorphism (41) of $\mathcal{D}(Y)$ to $\mathcal{D}(X)$ and, accordingly, the dual $C^\infty(X)$-homomorphism

$$\sigma' : \mathcal{D}(X)' \ni \xi \to \xi \circ \sigma \in \mathcal{D}(Y)'. \tag{48}$$

It follows that there is a $C^\infty(X)$-module monomorphism

$$Y^*(X) \underset{C^\infty(X)}{\otimes} \mathcal{D}(X)' \to \mathcal{D}(Y)' \tag{49}$$

such that

$$(\sigma \otimes \xi)(s) = \xi((s, \sigma)), \qquad s \in \mathcal{D}(Y), \qquad \sigma \in Y^*(X), \qquad \xi \in \mathcal{D}(X)'.$$

Theorem 14: The monomorphism (48) is a $C^\infty(X)$-module isomorphism

$$\mathcal{D}(Y)' = Y^*(X) \underset{C^\infty(X)}{\otimes} \mathcal{D}(X)'. \tag{50}$$

Proof: A vector bundle $Y \to X$ of fibre dimension m admits a finite atlas $\{(U_i, h_i), \rho_{ij}\}$, $i, j = 1, \ldots, k$, [5]. Given a smooth partition of unity $\{f_i\}$ subordinate to a cover $\{U_i\}$, let us put

$$l_i = f_i(f_1^2 + \cdots + f_k^2)^{-1/2}.$$

It is readily observed that $\{l_i^2\}$ also is a partition of unity subordinate to $\{U_i\}$. Then any section $s \in \mathcal{D}(Y)$ is represented by a tuple $(s_1, \ldots, s_k)$ of local $\mathbb{R}^m$-valued functions $s_i = h_i \circ s|_{U_i}$ which fulfil the relations

$$s_i = \sum_j \rho_{ij}(s_j) l_j^2. \tag{51}$$

Let us consider a topological vector space

$$\overset{mk}{\oplus}\,\mathcal{D}(X), \tag{52}$$

which also is a topological $C^\infty(X)$-module. There are both a continuous $C^\infty(X)$-monomorphism

$$\gamma : \mathcal{D}(Y) \ni s \to (l_1 s_1, \ldots, l_k s_k) \in \overset{mk}{\oplus}\,\mathcal{D}(X)$$

and a continuous $C^\infty(X)$-epimorphism

$$\Phi : \overset{mk}{\oplus}\,\mathcal{D}(X) \ni (t_1, \ldots, t_k) \to (\widetilde{s}_1, \ldots, \widetilde{s}_k) \in \mathcal{D}(Y), \qquad \widetilde{s}_i = \sum_j \rho_{ij}(l_j t_j). \tag{53}$$

In view of the relations (51),

$$\Phi \circ \gamma = \mathrm{Id}\,\mathcal{D}(Y),$$

and we have a decomposition

$$\begin{aligned} &\overset{mk}{\oplus}\,\mathcal{D}(X) = \gamma(\mathcal{D}(Y)) \oplus \mathrm{Ker}\,\Phi, \\ &t_i = [l_i \sum_j \rho_{ij}(l_j t_j)] + [t_i - l_i \sum_j \rho_{ij}(l_j t_j)], \end{aligned} \tag{54}$$

where $\gamma(\mathcal{D}(Y))$ consists of elements (t_i) satisfying the condition

$$t_i = l_i \sum_j \rho_{ij}(l_j t_j). \tag{55}$$

The topological dual of the topological vector space (52) is a $C^\infty(X)$-module

$$\overset{mk}{\oplus}\,\mathcal{D}(X)' \tag{56}$$

with elements $(\overline{t}_1, \ldots, \overline{t}_k)$. The epimorphism Φ (53) yields a $C^\infty(X)$-monomorphism

$$\Phi' : \mathcal{D}(Y)' \ni \xi \to \xi \circ \Phi \in \overset{mk}{\oplus}\,\mathcal{D}(X)'$$

such that $\Phi'(\mathcal{D}(Y)')$ vanishes on $\mathrm{Ker}\,\Phi$ in the decomposition (54). To describe $\Phi'(\mathcal{D}(Y)')$, let us consider the dual vector bundle $Y^* \to X$ provided with the conjugate atlas $\{(U_i, \overline{h}_i), \overline{\rho}_{ij}\}$ such that, for arbitrary sections s of $Y \to X$ and σ of $Y^* \to X$, the equality

$$(h_i \circ s, \overline{h}_i \circ \sigma)|_{U_i \cap U_j} = (\rho_{ij} \circ h_j \circ s, \overline{\rho}_{ij} \circ \overline{h}_j \circ \overline{s}) = (h_j \circ s, \overline{h}_j \circ \sigma)_{U_i \cap U_j} \tag{57}$$

holds. Then it is readily verified that the image $\Phi'(\mathcal{D}(Y)')$ of $\mathcal{D}(Y)'$ in the $C^\infty(X)$-module (56) consists of elements $(\overline{t}_i)$ satisfying the condition

$$\overline{t}_i = l_i \sum_j \overline{\rho}_{ij}(l_j t_j) \tag{58}$$

(cf. the condition (55)). This fact leads to the isomorphism (50).

Since Schwartz distributions on sections with compact support of a vector bundle $Y \to X$ constitute a $C^\infty(X)$ module $\mathcal{D}(Y)'$, differential operators on them can be introduced in accordance with Definition 1.

We restrict our consideration to $\mathcal{D}(Y)'$-valued differential operators on $\mathcal{D}(Y)'$. Of course, any multiplication (47) is a zero-order differential operator on $\mathcal{D}(Y)'$.

In accordance with Theorem 13, any $\mathcal{D}(Y)$-valued differential operator Δ on a $C^\infty(X)$-module $\mathcal{D}(Y)$ of sections with compact support is a continuous morphism of a topological vector space $\mathcal{D}(Y)$. Then it defines the dual morphism Δ' (46) of $\mathcal{D}(Y)'$.

Theorem 15: The transpose Δ' (46) of a k-order differential operator Δ on $\mathcal{D}(Y)$ is a differential operator on $\mathcal{D}(Y)'$ in accordance with Definition 1.

Proof: The proof is based on the fact that $\delta_f\Delta'$, $f \in C^\infty(X)$, (2) is the transpose of $-\delta_f\Delta$.

For instance, any connection ∇ (33) on $Y(X)$ and, consequently, on $\mathcal{D}(Y)$ define the transpose ∇'_u on $\mathcal{D}(Y)'$ for any vector field u on X. We have

$$< \phi, \nabla'_u(f\psi) >=< f\nabla_u(\phi), \psi >=< \nabla(f\phi), \psi > - < \mathbf{L}_u(f)\phi, \psi >=$$
$$< \phi, f\nabla'_u(\psi) > + < \phi, \mathbf{L}_{-u}(f)\psi > .$$

A glance at this equality shows that $-\nabla'_u$ is a connection on $\mathcal{D}(Y)'$.

In particular, let $Y = R$, and let $\mathbf{L}_u$, $u \in TX$, be the derivation (6) of $\mathcal{D}(X)$. Its transpose $\mathbf{L}'_u$ is called the Lie derivative of Schwartz distributions $\psi \in \mathcal{D}(X)'$ along u. In particular, if

$$\mathcal{D}(X)' \ni \psi = \overline{\psi} d^n x$$

is a density on X, then

$$\mathbf{L}'_u(\psi) = \mathbf{L}_{-u}(\psi) = -d(u\rfloor\psi) = -\partial_\mu(u^\mu\overline{\psi})d^n x.$$

It is a derivation of $\mathcal{D}(X)'$ because

$$\mathbf{L}'_u(f\psi) = \mathbf{L}_{-u}(f\psi) = \mathbf{L}_{-u}(f)\psi + f\mathbf{L}_{-u}(\psi).$$

The transpose Δ' (46) on $\mathcal{D}(Y)'$ of a differential operator Δ on $\mathcal{D}(Y)$ is continuous.

However, a differential operator on $\mathcal{D}(Y)'$ need not be the transpose of a differential operator on $\mathcal{D}(Y)$. Since $\mathcal{D}(Y)$ is reflexive and topologies on $\mathcal{D}(Y)$ and $\mathcal{D}(Y)'$ coincide with the weak ones, one can show the following.

Theorem 16: A differential operator on $\mathcal{D}(Y)'$ is the transpose of a differential operator on $\mathcal{D}(Y)$ iff it is continuous.

It follows that there is one-to-one correspondence between continuous differential operators on $\mathcal{D}(Y)'$ and differential operators on $\mathcal{D}(Y)$ whose module is isomorphic to

$$\mathrm{Diff}_k(\mathcal{D}(Y)) = ((J^kY)^* \underset{X}{\otimes} Y)(X) \tag{59}$$

in accordance with the isomorphism (44).

Jet modules $\mathcal{J}^k(\mathcal{D}(Y)')$ of a $C^\infty(X)$-module $\mathcal{D}(Y)'$ are introduced in accordance with Definition 3. By virtue of Theorem 4, any k-order differential operator Φ on $\mathcal{D}(Y)'$ is represented by a $C^\infty(X)$-homomorphism

$$\mathfrak{f}^\Phi : \mathcal{J}^k(\mathcal{D}(Y)') \to \mathcal{D}(Y)'. \tag{60}$$

Let Y^* be the dual of a vector bundle $Y \to X$. Then $Y^*(X) \subset \mathcal{D}(Y)'$ is a $C^\infty(X)$-submodule of $\mathcal{D}(Y)'$. It is easily verified that, if Δ is a differential operator on $\mathcal{D}(Y)$, its transpose restricted to $Y^*(X)$ takes its values into $Y^*(X)$. Therefore a differential operator Φ on $\mathcal{D}(Y)'$ is the transpose of a differential operator on $\mathcal{D}(Y)$ only if

$$\mathfrak{f}^\Phi(J^kY^*(X)) \subset Y^*(X), \tag{61}$$

i.e., the restriction

$$\Phi_* = \Phi|_{Y^*(X)} \tag{62}$$

of Φ is a $Y^*(X)$-valued differential operator on $Y^*(X)$. Let this condition hold. Then Φ is the transpose of a differential operator on $\mathcal{D}(Y)$ if it is the double transpose of Φ_* (62) as follows. Let Φ_* (62) be a $Y^*(X)$-valued differential operator on $Y^*(X) \subset \mathcal{D}(Y)'$ and, consequently, on a $C^\infty(X)$-module $\mathcal{D}(Y^*)$. Let Φ'_* be its transpose on $\mathcal{D}(Y^*)'$. Then the restriction of Φ'_* to $\mathcal{D}(Y) \subset \mathcal{D}(Y^*)'$ is a $\mathcal{D}(Y)$-valued differential operator on $\mathcal{D}(Y)$. Let Φ''_* be its transpose on $\mathcal{D}(Y)'$. If $\Phi \neq \Phi''_*$, then $\Phi - \Phi''_*$ is a non-zero differential operator vanishing on $\mathcal{D}(Y^*)$. Such a differential operator fails to be the transpose of a differential operator on $\mathcal{D}(Y)$ which must be neither non-zero because $\Phi - \Phi''_*$ vanishes on $\mathcal{D}(Y^*)$ nor zero because $\Phi - \Phi''_*$ is non-zero.

Example 3: Let us consider a space $\mathcal{D}(X)$ of test functions on a manifold X. A vector space of Schwartz distributions $\mathcal{D}(X)'$ is a free $\mathbb{R}$-module whose basis B contains the Dirac measure ϵ_x at a point $x \in X$. Let us consider a vector subspace $\mathcal{D}(X)'_x \subset \mathcal{D}(X)'$ whose basis is $B \setminus \epsilon_x$. A vector subspace $\mathcal{D}(X)'_x$ also is a $C^\infty(X)$-submodule of $\mathcal{D}(X)'$ because $f\epsilon_x = f(x)\epsilon_x$, $f \in C^\infty(X)$. Accordingly, an $\mathbb{R}$-epimorphism

$$\gamma : \mathcal{D}(X)' \to \mathcal{D}(X)'_x, \qquad \gamma(\mathcal{D}(X)'_x) = \mathrm{Id}\,\mathcal{D}(X)'_x, \qquad \gamma(\epsilon_x) = 0,$$

also is a $C^\infty(X)$-homomorphism, i.e., a zero order differential operator on $\mathcal{D}(X)'$. It is not the transpose of a differential operator on $\mathcal{D}(X)$.

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